The role of aging in a minimal model of earthquakes

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We introduce a simple model of earthquakes, inspired by the spring-block models, but describing contacts at a mesoscale. A single contact point synthesizes many rock contacts so that these "macrocontacts" can have an internal dynamics, described by a stochastic process, that leads to an evolution of their breaking thresholds. This aging process leads to the Gutenberg-Richter law, which relates the probability of occurrence of earthquakes to their magnitude. An analytical approach is used to determine the range of magnitudes in which this law applies.

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I. INTRODUCTION

Although the modeling of earthquakes is certainly still a goal that is very far away, for a long time physicists have been developing simple models that could reproduce their main features, in an attempt to understand the origin of the observed "laws." One of the few empirically established laws in seismology is the Gutenberg-Richter (GR) law [1], which states that the number of earthquakes (EQs) with a magnitude $\geqslant \mathcal{M}$ scales with $\mathcal{M}$ as

$$\mathcal{N}(\mathcal{M}) \propto 10^{-b\mathcal{M}} \propto \mathcal{A}^{-2b/3}, \tag{1}$$

where the dimensionless EQ magnitude is defined as

$$\mathcal{M} = \tfrac{2}{3}\log_{10}(\mathcal{A}/\mathcal{A}_0). \tag{2}$$

In this equation $\mathcal{A}$ is the EQ amplitude (event size, energy emitted, stress relaxed, stress drop, etc.), $\mathcal{A}_0$ is a constant, and $b = 1$ (or $b = 0.5 - 1.5$ in a general case [2,3]) is a numerical constant. The distribution of EQ magnitudes $\mathcal{P}_{\mathcal{M}}(\mathcal{M})$ may then be defined by $\mathcal{N}(\mathcal{M}) = \int_{\mathcal{M}}^{\infty} d\mathcal{M}'\,\mathcal{P}_{\mathcal{M}}(\mathcal{M}')$, which gives $\mathcal{P}_{\mathcal{M}}(\mathcal{M}) = -d\mathcal{N}(\mathcal{M})/d\mathcal{M} \propto 10^{-b\mathcal{M}} \propto \mathcal{A}^{-2b/3}$. If we introduce the distribution of EQ amplitudes $\mathcal{P}(\mathcal{A})$ defined by the relationship $\mathcal{P}(\mathcal{A})\,d\mathcal{A} = \mathcal{P}_{\mathcal{M}}(\mathcal{M})\,d\mathcal{M}$, then from Eq. (1) we obtain

$$\mathcal{P}(\mathcal{A}) \propto \mathcal{A}^{-\nu} \quad \text{with} \quad \nu = 1 + 2b/3, \tag{3}$$

where $\nu = 5/3 \approx 1.67$ (or $\nu = 1.33 - 2$ in a general case). To normalize the distribution $\mathcal{P}(\mathcal{A})$, one has to introduce a minimal EQ size $\mathcal{A}_{\min}$; then $\mathcal{P}(\mathcal{A}) = (2b/3)\mathcal{A}_{\min}^{2b/3}\mathcal{A}^{-\nu}$ gives $\int_{\mathcal{A}_{\min}}^{\infty} d\mathcal{A}\,\mathcal{P}(\mathcal{A}) = 1$. Notice that the average size of EQs for the distribution (3) is infinite, $\langle\mathcal{A}\rangle = \int_{\mathcal{A}_{\min}}^{\infty} d\mathcal{A}\,\mathcal{A}\,\mathcal{P}(\mathcal{A}) = \infty$ if $\nu < 2$ (or $b < 1.5$).

The hyperbolic law (1), in contrast to the exponential decay typical for most phenomena in physics, indicates a long-lived nature of the EQ activity or some self-organized phenomenon. The first physical model of EQs was proposed by Burridge and Knopoff [4], the well-known BK spring-block model, and further developed in many studies (e.g., see Refs. [5–13]). In these models, two plates are coupled by a set of contacts which deform when the plates move relatively to one another. The contacts are frictional; i.e., they behave as elastic springs as long as their stresses are below some threshold f_s and break and reattach in a less-stressed state when the threshold is exceeded. The EQ amplitude $\mathcal{A}$ is typically associated with the number of broken contacts s at an event. Typically these models demonstrate a GR-like power-law behavior,

$$\mathcal{P}_h(s) \propto s^{-\nu}\phi(hs), \tag{4}$$

but only for a restricted interval of amplitudes, $s \ll s_m = h^{-1}$ [here $\phi(x)$ is a scaling function describing the shape of the cutoff, $\phi(x) \to 1$ for $x \ll 1$ and $\phi(x)$ decays exponentially for $x \gg 1$]. The Olami-Feder-Christensen model (OFC) is a variant of the BK model [7,8], where the contacts are organized in a square lattice. When a contact breaks, a part αf_s of its stress is equally redistributed over four nearest neighbors; for $\alpha = 1/4$ the model is "conservative." This model does demonstrate the power law (4) for a wide interval of EQs amplitudes [the value s_m grows linearly (or slower) with the system size N [9,10]]. The exponent ν depends on the parameter α; the GR value $b = 1$ is observed for $\alpha \approx 0.2$.

In spite of its interesting features, the OFC model does not bring a satisfactory answer to the question of finding a simple model reproducing the GR law for two reasons: (1) a model where all the contact thresholds are the same is somehow artificial; when the interactions between the contacts is absent, $\alpha = 0$, this model is "singular" and admits periodic solutions only [14,15]; and (2) the power law emerges only for the variant with open boundary conditions [9,10], when avalanches emerge at a boundary due to spatial inhomogeneity (boundary sites have fewer nearest neighbors) and propagate into the interface (as a result, the transient time to reach the steady regime with the power-law distribution of EQs goes to infinity with increasing system size [10]).

A more realistic model is obtained if each contact is characterized by its own threshold f_{si}, with a continuous distribution $P_c(f_{si})$ [11,13–15]. Typically one assumes that $P_c(f_{si})$ is Gaussian with an average f_s and a standard deviation δf_s. These models may also exhibit a power law (4), but now s_m is determined by model parameters and remains finite when the system size increases. However, the interval of $\mathcal{M}$

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values where the power law is valid in simulations is typically $\Delta\mathcal{M} \lesssim 2$ (e.g., $\Delta\mathcal{M} \approx 2.2$ in Ref. [12], 2.1 in Ref. [13], and 2.3 in Ref. [11]), i.e., much smaller than that observed in real EQs, where $\Delta\mathcal{M} > 6$ [16]. Serino *et al.* [11] proposed to consider an ensemble of OFC-like models having different parameters and moving independently; then the average distribution $\mathcal{P}(s) = \int dh\, \mathcal{D}(h)\mathcal{P}_h(s)$ with an appropriate weighting function $\mathcal{D}(h)$ may lead to the desired GR power law with $b = 1$. One possible extension of these block-springs models is to incorporate a slow relaxation of the stress at contacts to account for their plastic deformation [12,13].

On the other hand, EQ-like models appeared rather successful in recent studies of friction [17–23]. MD simulation showed that the frictional interface is generally not uniform, its different regions being characterized by different stress thresholds for the onset of sliding. Such a situation is naturally described by the EQ-like model with a distribution of static-friction thresholds. Moreover, kinetics of the EQ model may be reduced to a master equation (ME) which allows an analytical study [14,15], in contrast to the EQ model which, being a cellular-automaton type model, has to be numerically investigated. This approach describes the stick-slip motion [15,21] and may be generalized to include temperature effects and the aging of the contacts (frictional strengthening of the interface due to, e.g., growth of ths contacts' sizes, or their reconstruction, chemical "cementation", etc.), leading to a dependence of the friction force on the velocity v. When an interaction between the contacts is taken into account [22], the EQ model of friction describes the onset of sliding, demonstrating different detachment fronts and precursors [23] in agreement with experiments. Considering real EQs, these precursors could be interpreted as the fore-shocks observed in EQs.

The master equation approach naturally introduces a concept of "macrocontact," which, instead of an individual contact, describes a group of contacts through its effective properties resulting from statistical behaviors of many individual contacts. Such macrocontacts may therefore have properties which are much richer than individual ones and moreover may depend on time as the dynamics of individual contacts modify their collective properties. This introduces the idea that contacts could have time-dependent properties, which would be useful at the scale of EQs too. In this context, considering the frictional contacts as "static" ones having fixed parameters such as thresholds and stiffness can be questioned. This may be acceptable for the frictional interface where the plates move with a speed up to m/s, but is unlikely to be true for EQs where the typical velocity of a tectonic plate is $v \sim 30$ mm/year $\approx 3 \times 10^{-9}$ m/s. Here we propose a EQ-like model with "living" contacts, where the contact parameters are not fixed but evolve with time from event to event.

In this paper we introduce a model in which the "contacts" actually describe the interaction of two tectonic plates at a rather large scale, i.e., must be regarded as "macrocontacts." An EQ can be the breaking of a single of these macrocontacts, although, if they are interacting it may also involve several of them. We show that, if one describes the aging of those contacts as an independent time-dependent statistical process, this process can be responsible for a Gutenberg-Richter law for the statistics of the events. We then show how the master equation approach applied to this extended model can be used to evaluate the largest magnitude for which the model can lead to the GR law.

II. MODEL

A. Basic features

The model that we consider starts from the same premises as block-spring models such as the OFC model and its variants. We consider a tectonic plate that slides with a velocity v over another plate, called the base, which is taken as the reference point assumed to be fixed. The two plates are coupled by an array of "contacts," but, as explained in the introduction, those contacts should be viewed as describing a larger scale than the contacts in OFC models because they synthesize in a single connection the many interactions that occur between the rocks in a region of a fault. To give an order of magnitude, one can consider that a "contact" represents the interaction of the two sides of a fault on a scale of a few kilometers, while the actual "microcontacts" of the rocks take place at a scale which can be of the order of millimeters to meters. Henceforth we will only use the simple term of contact, but the collective aspect of such a contact should be kept in mind.

In the spirit of the spring-blocks models, we consider a two-dimensional array of contacts organized in a triangular lattice of size N and lattice spacing a with periodic boundary conditions. Each contact is characterized by a shear force $f_i = k_i l_i$, where k_i is the contact stiffness and l_i is its stretching. Each contact is also characterized by a threshold value f_{si}; the ith contact is elastic until $|f_i| < f_{si}$ but breaks when the stress exceeds the threshold. We assume that the thresholds f_{si} have a continuous distribution $P_c(f)$; the elastic constants k_i also have a continuous distribution which is coupled with the distribution $P_c(f)$ as $k_i = k\,(f_{si}/f_s)^{1/2}$ (see Refs. [14,15]). When a contact breaks at $f_i = f_{si}$, it takes a value $f_i \sim 0$, and a new value for its threshold is assigned. The (rigid) plate moves by ΔX per time step $\Delta t = \Delta X/v$ (in contrast to the cellular automaton algorithm, here we assume that ΔX is fixed; see Sec. V), and all contact stretchings get the increment ΔX per time step.

Our numerical algorithm is organized in the following way. At each time step we look for the largest value of $f_i - f_{si}$, and, if it is $\geqslant 0$, the ith contact breaks so that the total force on it (together with the contribution due to interaction with the neighboring contacts when it is taken into account) becomes equal to zero; i.e., the contact is restored in the "zero force position" as proposed by Olami *et al.* [7] (note that the "zero stretching position" algorithm [9,10,12], where $l_i \to 0$ at the break event, as well as the rule $l_i \to l_i - 1$ used in Ref. [13], may deeply modify the results). Then, at the same time step, we recalculate all Δf_i values and again look for the largest one, repeating the process until all forces fall below the thresholds that break contacts, and calculate the corresponding drop of the force which is applied on the sliding plate by the contacts, ΔF_i, at the given time step.

B. Elastic interaction

Elasticity of the sliding tectonic plates can lead to an interaction between the contacts: When one of the contacts breaks, the shear stress sustained by this contact must be redistributed among the neighboring contacts and may even cause an avalanche of contacts breaking [22]. To incorporate this effect, we assume that because of the elastic interaction between the contacts i and j, the forces acting on these contacts have to be corrected as $f_i \to f_i - \Delta f_{ij}$ and $f_j \to f_j + \Delta f_{ij}$, where $\Delta f_{ij} = k_{ij}\,(l_j - l_i)$ in the linear approximation. For example, let at $t = 0$ the contacts be relaxed, $l_i(0) = l_j(0) = 0$, and then, due to plate motion, all stretchings grow together, so that still $\Delta f_{ij} = 0$. Then, let at some moment t the jth contact break, $l_j(t) \to 0$, while the ith contact is still stretched, $l_i(t) > 0$, so that $\Delta f_{ij}(t) = -k_{ij}\,l_i(t) < 0$. Therefore, when the jth contact breaks, the force on the ith contact increases.

To incorporate the interaction between the contacts in our model, we assume that the stretching of the given (ith) contact has an additional contribution $\Delta f_i = \sum_{j\,(j\neq i)} k_{ij}(l_j - l_i)$, where the sum is over the neighboring contacts inside some circle of the radius r_c (in simulation we used $r_c = 3a$), and $k_{ij} \propto r_{ij}^{-3}$; we put $k_{ij} = \kappa k (a/r_{ij})^3$, where the dimensionless parameter κ (which is an analog of the parameter α in the OFC model [7]) characterizes the strength of interaction.

C. Aging of contacts

While the other features of our model are similar to earlier studies [7–13], the description of aging is the important new feature of our approach. Besides the dynamics of the cellular automaton, we introduce a second dynamics for the properties of the contacts, which is particularly important because, as discussed above, in this model the contacts synthesize many actual rock contacts and can therefore have their own dynamics.

When a contact is formed, it appears with a threshold $f_{si}(t=0) \sim 0$ (in simulations we assumed that the newborn contacts emerge with a threshold f_{si} taken from the Gaussian distribution with the average $f_{s\,\text{newborn}} = 0.01 f_s$ and deviation $\delta f_{s\,\text{newborn}} = f_{s\,\text{newborn}}$, but when $f_{si} \leqslant 0$, a new value is called). Then the model describes the growth of the threshold versus time, due to aging. The peculiarity of our model is that it describes the evolution $f_{si}(t)$ as a stochastic process. This is associated to the fact that, as discussed above, at the scale appropriate to describe an EQ the contacts are actually "macrocontacts" representing a large number of rock interactions. Each of the local contacts ("microcontacts") evolves due to plastic deformations or the slow formation of chemical bonds, due, for instance, to the effect of the ground water, and these local events, occurring randomly, gradually modify the global properties of a macrocontact. One can describe the collective behavior at the scale of a macrocontact by a stochastic equation, in the same spirit as the Langevin equation describes the evolution of the position of a Brownian particle (macrovariable) under the influence of multiple collisions with the molecules of a fluid. During a time interval dt the variation of the threshold f_{si} is the sum of two contributions $df_{si} = K(f_{si})\,dt + G\,dw$. The first contribution $K(f_{si})\,dt$ is the drift term, which describes the systematic effect of the individual events, analogous to the friction term in Brownian motion. The second contribution $G\,dw$, where w is a random process (Wiener process), corresponds to random fluctuations, with zero average $\langle dw\rangle = 0$ and a vanishingly short correlation $\langle dw\,dw\rangle = dt$ [24]. The evolution of $f_{si}(t)$ can also be formulated as a Langevin equation

$$df_{si}(t)/dt = K(f_{si}) + G\,\xi(t), \tag{5}$$

where $\xi(t)$ is a Gaussian random force, $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\,\xi(t')\rangle = \delta(t-t')$. This stochastic equation is equivalent to the Fokker-Planck equation (FPE) for the distribution of thresholds $P_c(f_{si};t)$:

$$\frac{\partial P_c}{\partial t} + \frac{dK}{df_{si}} P_c + K\frac{\partial P_c}{\partial f_{si}} = \frac{1}{2}\,G^2\frac{\partial^2 P_c}{\partial f_{si}^2}. \tag{6}$$

The choice of the drift and fluctuating terms is guided by the expected properties of the contacts. We assume that the drift force is given by the expression

$$K(f_{si}) = \left(\frac{f_s}{\tau_0}\right)\beta^2\frac{1 - f_{si}/f_s}{1 + \varepsilon(f_{si}/f_s)^2}, \tag{7}$$

where f_s is a model parameter, while the amplitude of the stochastic force is equal to

$$G = \beta\delta f_s\sqrt{2/\tau_0}, \tag{8}$$

where β and ε are two dimensionless parameters. With this choice, in the case of $\varepsilon = 0$, the stationary solution $P_{c0}(f)$ of the FPE corresponds to the Gaussian distribution $P_{c0}(f) = (2\pi)^{-1/2}(\delta f_s)^{-1}\exp[-\frac{1}{2}(1 - f/f_s)^2/\delta^2]$, where δ and τ_0 are parameters. The relation (8), required from the validity of this limit, expresses the fact that the systematic and random contributions actually come from the same origin, the individual fluctuations of the local contacts. It is the equivalent of the fluctuation-dissipation relation for the Brownian motion. The ε contribution in Eq. (7) is a simple way to ensure aging with a growth of the threshold. For $\varepsilon\delta^2 < 0.5$ the average threshold tends to saturate to a finite value while for $\varepsilon\delta^2 > 0.5$ it keeps growing. Moreover for $\varepsilon > 0$ the distribution of thresholds is no longer Gaussian and acquires a power tail, $P_{c0}(f) \propto f^{-1/\varepsilon\delta^2}$ for $f \gg f_s$.

Thus, the dimensionless parameter ε determines the deviation of the distribution of thresholds from a Gaussian one, while the parameter β corresponds to the rate of aging, which should be compared with the driving velocity set to $v = 1$.

D. Relaxation of contacts

Finally, we can also incorporate a slow relaxation of contacts (e.g., because of transient material creep) in order to allow for spatiotemporal correlations between the events. We assume that at every time step (after checking and breaking the contacts), all thresholds are "relaxed" according to a "diffusion" law ($\vec{r}$ is the 2D vector in the interface):

$$\frac{\partial f_s(\vec{r})}{\partial t} = \widetilde{D}\,\nabla_{\vec{r}}^2 f_s(\vec{r}). \tag{9}$$

This implies that the local stress is slowly transferred to the neighboring sites. For the discrete lattice this equation takes the form $f_{is} \to f_{is} + \widetilde{D}\,(\nabla_{\vec{r}}^2 f_{is})\,\Delta t$; for the triangular lattice for the site $i = 0$, omitting the index s, we have $(\nabla_{\vec{r}}^2 f_0) = (f_1 + f_2 - 2f_0)/a_x^2 + (f_3 + f_4 + f_5 + f_6 - 4f_0)/(a_x/2)^2 + (f_3 + f_4 + f_5 + f_6 - 4f_0)/a_y^2$, where $a_x = a$ and $a_y = a\sqrt{3}/2$, the contacts "1" and "2" are the nearest neighbor ones of the contact "0" along x (the driving direction) and the contacts "3" to "6" are the up or down nearest neighbors.

E. Parameters

Four parameters of our model may be fixed without loss of generality; we put $a = 1$, $v = 1$, $f_s = 1$, and $k = 1$. Then, the natural unit of length is $a_0 = f_s/k = 1$ and the natural time unit is $\tau_0 = a_0/v = 1$. The parameter $\delta = \delta f_s/f_s$ describes the thresholds dispersion; in what follows we put $\delta = 1$. Thus, our model is characterized by four dimensionless parameters:

(i) ε describes a deviation of the threshold distribution $P_c(f)$ from the Gaussian one,

(ii) β describes the rate of aging (in our units the limit $v \to 0$ corresponds to $\beta \to \infty$),

(iii) κ describes the strength of interaction, and

(iv) $D = \widetilde{D}/(a^2/\tau_0)$ describes the rate of relaxation.

III. SIMULATION RESULTS

We define the (dimensionless) amplitude of a EQ as the sum of the drops of forces on contacts at every time step, $\mathcal{A} = \sum_i \Delta F_i/f_s$. In simulation we typically used the lattice $N = 30 \times 34 = 1020$ so that $\mathcal{M}_{\max} = \frac{2}{3}\log_{10} N \approx 2$. In the initial state, all contacts are relaxed, $f_i = 0$. The simulation time was $t_{\max} = 10^3\,\tau_0$ or longer; the first $\gtrsim 10\%$ of data were discarded to reach the steady state. The displacement step was $\Delta X = 10^{-4} a_0$, but we checked how a change of ΔX may disturb the results (see Sec. V). The results are presented in Figs. 1 and 2.

Figure 1 shows the results in the absence of interaction between the contacts ($\kappa = 0$) and without relaxation ($D = 0$), but in the presence of aging, determined by the value of ε. For $\varepsilon = 0$ (limit of Gaussian distribution of thresholds) the statistics of the events is a simple exponential law. As soon as $\varepsilon > 0$ it becomes a power law for an interval $\Delta\mathcal{M}$ of magnitudes, and the exponent b decreases when ε increases; the GR value $b \approx 1$ is achieved for $\varepsilon \approx 0.5$. In considering these results one should keep in mind the difference between our model and the models deriving from the OFC approach. As we are here examining the situation where $\kappa = 0$, an "event" is actually the breaking of a single (macro-)contact between the plates (or very few contact breakings, as discussed in Sec. V). This can nevertheless lead to nontrivial statistics because the properties of the contacts are themselves nontrivial and resulting from the stochastic process Eq. (5).

Figure 1(b) shows that the power-law interval $\Delta\mathcal{M}$ increases with β: The slower the plate moves, the larger is the interval of EQ magnitudes where the GR law operates.

Figure 2 demonstrates the role of interaction ($\kappa > 0$) and relaxation ($D > 0$). Interactions and relaxation of the contacts have qualitatively similar effects, they tend to kill large EQs while they significantly increase the number of small events, which exceeds the number expected from the GR law. For EQs this would correspond to events that occur in several places quasisimultaneously, a small EQ in one place (breaking of a contact) leading to another small one nearby, triggered by the interaction. In other words the interactions tends to favor the relaxation of the tensions before a very large event has time to build up. On the other hand very large EQs can be expected to

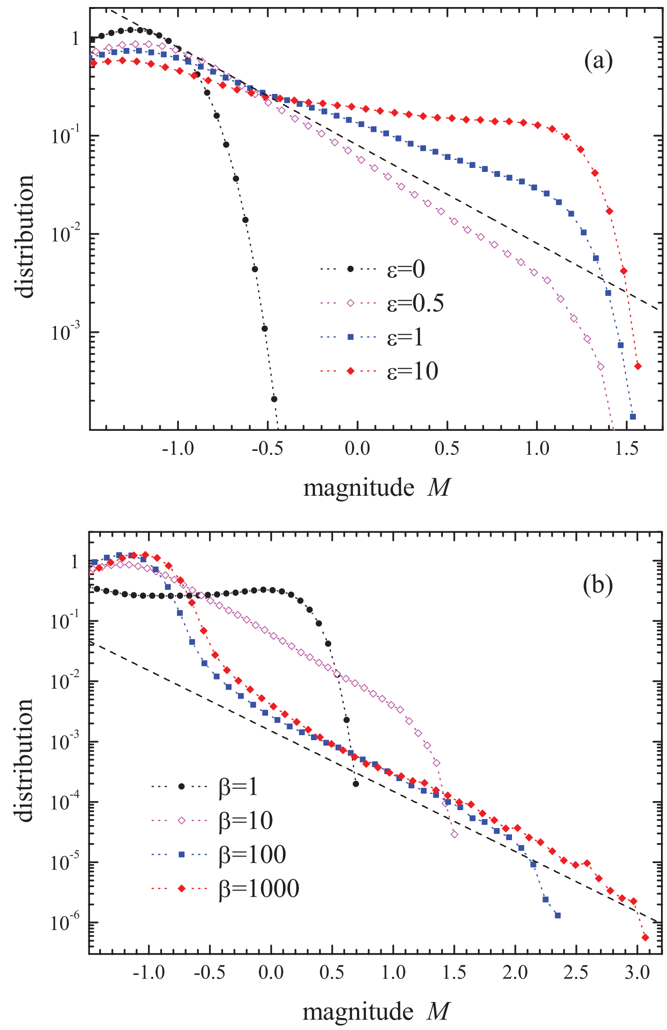

FIG. 1. (Color online) The distributions $\mathcal{P}(\mathcal{M})$ for $\kappa = 0$, $D = 0$, $N = 1020$: (a) for $\beta = 10$ and different values of ε ($t_{\max} = 10^3$), (b) for $\varepsilon = 0.5$ and different values of β ($t_{\max} = 10^4$). The line shows the GR power law with $b = 1$.

have a very localized epicenter while small tremors could be hard to localize precisely.

It is interesting to notice that the largest domain of validity of the GR law is obtained for $\varepsilon = 0.5$, which is the largest value of ε for which the aging of the contacts leads to a deviation from a Gaussian distribution without leading to an indefinite growth of the average threshold.

IV. MASTER EQUATION

The results plotted in Fig. 1 show that, in the presence of aging, this model is able to exhibit a GR law for the statistics of the events. As discussed in the introduction, the question posed by all simplified models of EQs is the range of validity of this law, which is generally found to be limited, not only by the size of the model system but rather by fundamental limits of the model. To examine this aspect for our model with aging, rather than studying the evolution of the EQ model by numerical simulation, it is possible to describe it analytically [14,15].

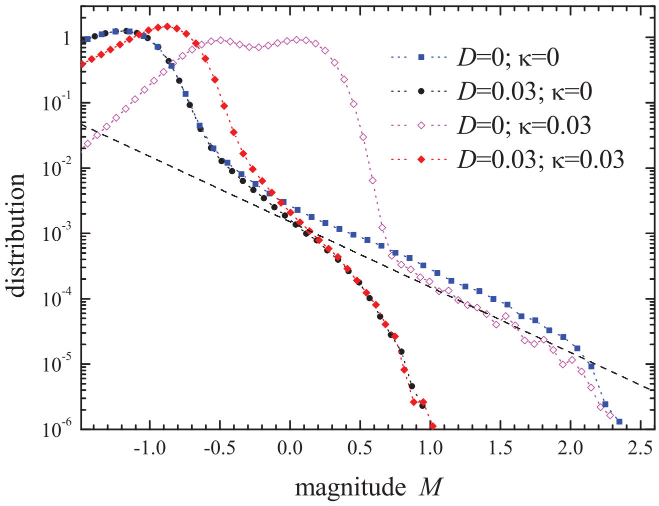

FIG. 2. (Color online) The distributions $\mathcal{P}(\mathcal{M})$ for $\varepsilon = 0.5$, $\beta = 100$ and different values of κ and D ($N = 1020$, $t_{\max} = 5 \times 10^3$). Line shows the GR power law with $b = 1$.

Let us consider the simplest model which gives the GR law, i.e., ignore the interaction between the contacts as well as their relaxation, $\kappa = D = 0$. Let $P_{cx}(x)$ be the normalized probability distribution of values of the stretching thresholds x_{si} at which contacts break; it is coupled with the distribution of threshold forces by the relationship $P_{cx}(x)\,dx = P_c(f)\,df$. With the simplified assumption that $\kappa = D = 0$, i.e., when EQ events correspond to the breaking of a single contact, $P_{cx}(x)$ determines the statistics of the EQs.

To describe the evolution of the model, we introduce the distribution $Q(x;X)$ of the stretchings x_i when the bottom of the sliding plate is at a position X. This amounts to looking at the system at an even larger scale. We have already introduced "contacts" that actually describe the collective behavior of many rock contacts. The statistical distributions that we introduce here are statistics over many of these "macrocontacts," therefore they should be viewed as statistics of the events at the scale of a fault. At this scale the evolution of the system is described by the master equation [14,15]

$$\left[\frac{\partial}{\partial X} + \frac{\partial}{\partial x} + P(x)\right] Q(x;X) = R(x)\,\Gamma(X), \tag{10}$$

where

$$\Gamma(X) = \int_{-\infty}^{\infty} d\xi\; P(\xi)\, Q(\xi;X) \tag{11}$$

and $P(x)\,\Delta X$ is the fraction of contacts that break at the stretching x when the plate moves by ΔX. It is related to the distribution of the breaking thresholds $P_{cx}(x)$ by

$$P(x) = P_{cx}(x)/J_c(x), \quad J_c(x) = \int_x^{\infty} d\xi\; P_{cx}(\xi), \tag{12}$$

which simply says that the fraction of the contacts that break when X increases by ΔX are those that have their thresholds between x and $x + \Delta X$ divided by the total of fraction of contacts which are not yet broken at stretching x, given by $J_c(x)$.

The steady state corresponds to the average properties of the fault on the long term. It corresponds therefore to the statistics of the EQs observed over centuries at a given fault. In the steady state the ME reduces to

$$dQ(x)/dx + P(x)\,Q(x) = R(x)\,\Gamma. \tag{13}$$

The solution of this equation in the absence of the right-hand side is of the form

$$Q(x) \propto \exp[-U(x)], \quad U(x) = \int_0^x d\xi\; P(\xi), \tag{14}$$

i.e.,

$$Q(x) \propto J_c(x), \tag{15}$$

as one can easily check from the relation (12). In the presence of the right-hand side the solution is

$$Q_s(x) = \Gamma\; J_c(x) \int_{0^-}^{x} d\xi\; R(\xi)/J_c(\xi), \tag{16}$$

where Γ is the normalization constant determined by $\int_0^\infty dx\; Q_s(x) = 1$. In the case of $R(x) = \delta(x)$ Eq. (16) reduces to $Q(x) = \Theta(x) J_c(x)/C[P]$, where $\Theta(x)$ is the Heaviside step function [$\Theta(x) = 1$ for $x \geqslant 0$ and 0 otherwise], and $C[P] = \int_0^\infty dx\; J_c(x)$. Note also that in the steady state

$$\Gamma = 1/C[P], \tag{17}$$

because $\int_0^\infty d\xi\; P(\xi) J_c(\xi) = \int_0^\infty dU\; e^{-U} = 1$.

This formal solution for $Q_s(x)$ is not yet the full solution of the problem because $P_{cx}(x)$ is unknown. It should result from the aging of the contacts. In Sec. II C we introduced Eq. (6), which tells us how the distribution of the thresholds evolves *under the effect of aging alone*. This equation may be rewritten as

$$\frac{\partial P_c}{\partial t} = \frac{\beta^2}{\tau_0}\,\hat{L}\,P_c, \quad \hat{L} = \frac{\partial}{\partial \phi}\left(\frac{\phi - 1}{1 + \varepsilon\phi^2} + \delta^2 \frac{\partial}{\partial \phi}\right), \tag{18}$$

where $\phi = f/f_s$. However, because the contacts continuously break and form again when the plate moves, this introduces two extra contributions in the equation determining $\partial P_c/\partial t$ in addition to the pure aging effect described by Eq. (18): A term $P(x;X)\,Q(x;X)$ takes into account the contacts that break, while their reappearance with the threshold distribution $P_{ci}(x)$ [$P_{ci}(x) = \delta(x)$ in our model] gives rise to the second extra term in the equation. Therefore, the full evolution of P_{cx} is described by the equation

$$\begin{aligned} &\frac{\partial P_{cx}(x;X)}{\partial X} - \frac{\beta^2}{v\tau_0}\widehat{L}\,P_{cx}(x;X) + P(x;X)\,Q(x;X) \\ &\quad = P_{ci}(x)\,\Gamma(X). \end{aligned} \tag{19}$$

Thus, we come to the set of equations (10)–(12) and (19). For the steady-state regime in the case of $R(x) = \delta(x)$, Eq. (19) reduces to

$$\beta^2 C[P]\,\widehat{L}\,P_{cx}(x) = v\tau_0 P(x)\,J_c(x), \quad x > 0, \tag{20}$$

where we used Eqs. (16) and (17). Taking also into account the identity $P(x)\,J_c(x) = P_{cx}(x)$, we finally come to the equation

$$\beta^2 C[P]\,\widehat{L}\,P_{cx}(x) = v\tau_0 P_{cx}(x), \quad x > 0. \tag{21}$$

Equation (21) is equivalent to the integral equation [25]

$$P_{cx}(x) = P_{c0}(x)\left[1 - \lambda \int_{0^+}^{x} d\xi\, J_c(\xi) P_{c0}^{-1}(\xi)\right], \tag{22}$$

where $\lambda = v\tau_0/\beta^2 a_0^2 C[P]$ and $P_{c0}(x)$ is the distribution of stretchings due to aging alone, introduced in Sec. II C. And, as mentioned above, for noninteracting contacts $P_{cx}(x)$ is also the distribution of the events in the steady state of a fault. Equation (22) may be solved by iterations in the case of $\lambda \ll 1$. For $\varepsilon > 0$, i.e., in the presence of aging, $P_{c0}(x)$ has a power-law tail $P_{c0}(x) \propto x^{-\nu}$. We have $J_{c0}(x) \propto (\nu-1)^{-1} x^{-(\nu-1)}$. The GR-like power law of the events survives as long as the correction term [second term in the bracket in Eq. (22)] stays small enough. This gives us the limit of validity of the GR law that we were looking for. The GR-like power law should operate for EQ amplitudes lower than

$$f_{\max} \sim \beta f_s \sqrt{(\nu - 1)C[P]/v\tau_0} \tag{23}$$

or

$$\mathcal{M}_{\max} \sim \tfrac{2}{3}\log_{10}(\beta/\sqrt{v}), \tag{24}$$

which can be tailored to the desired value by the choice of parameters, provided the model is large enough.

V. DISCUSSION

When we discuss the GR law in terms of $P_{cx}(x)$ we assume that an EQ is associated to the breaking of a single contact; i.e., we put $\mathcal{A} = f$. This is legitimate because we have considered the case $\kappa = 0$, i.e., we do not take into account the interaction between the macrocontacts. Therefore a breaking does not *induce* another breaking in its vicinity. However, this does not mean that, in our algorithm where the motion of the plates occurs in steps ΔX, two events cannot occur fortuitously at the same step. In the cellular automaton algorithm, where every EQ event is treated separately (this corresponds to the MD algorithm in the limit $\Delta X \to 0$), the EQ distribution is equal to

$$\mathcal{P}_1(f) \propto (P_c \otimes R)(f), \tag{25}$$

where the convolution of two functions is defined as

$$(A \otimes B)(x) \equiv \int d\xi\, A(x-\xi)\, B(\xi). \tag{26}$$

When the newborn contacts appear with zero stretching, $R(f) = \delta(f)$, we have $\mathcal{P}_1(f) = P_c(f)$.

However, in the MD algorithm with a finite step ΔX, there is a nonzero probability that two events may occur during the same step; the distribution of such events is $\mathcal{P}_2(f) \propto (\mathcal{P}_1 \otimes \mathcal{P}_1)(f)$, and so on for the occurrence of several events during the same step. This could affect the statistics that we measure in the simulations. However, this effect is mostly likely to occur for distributions $P_{cx}(x)$ which are highly peaked and tend to concentrate the events. For the case of interest for EQs, where the distribution has a tail at large x, the probability of quasisimultaneous occurrence of events is low, and the simulation determines $P_{cx}(x)$ with a good accuracy when ΔX is small enough. We have tested this by performing simulations with different steps. The step $\Delta X = 10^{-4}$ has been used for the results presented in the paper. It corresponds to a time interval $\Delta t = \Delta X/v$. With $v = 1$ and events observed with a time interval of a few units in the calculations, if we consider that a real EQ has a duration of the order of a minute, it sets the time scale for the value $\Delta t = 10^{-4}$ in the calculations. Events separated by a few time units in the calculation are separated by a few weeks in actual time, which is a realistic scale validating the choice $\Delta X = 10^{-4}$.

The main result of this work is that the GR law for EQs can be generated by the *aging of the contacts alone* since it can be observed even in the absence of interaction or relaxation. This is possible in such a model because a "contact" is actually describing many rock interactions and has its own dynamics. Of course the model could be completed in many ways to approach more realistic situations. We showed here that interaction and relaxation of the contacts tend to reduce the domain in which the GR law applies, essentially because they tend to facilitate the occurrence of small events that relax the system. One thing that is missing in such a description is the elasticity of the plates which interact at the fault. Here they are rigid so that the displacement ΔX at each step is the same for all contacts. The elasticity effect is probably important in the case of friction. Perhaps for the large scale of the tectonic plates it plays a smaller role. Equation (24) shows that the largest EQ above which one could expect the GR law to break down is determined by the velocity v of the moving tectonic plate and β which measures the rate of aging. Therefore, as these quantities can be expected to be roughly constant for given tectonic plates, statistics on one region could be used to infer the risks in another region belonging to the same plate.

The process describing the aging of the contacts needs further investigations as it is presently largely arbitrary. We use a stochastic process that goes to a Gaussian distribution in one limit and a power law in another, but how this process can be generated by the dynamics of "macrocontacts" is still open. A local OFC-like model describing a macro-contact could generate such a power law, although not as a stochastic process. What is nevertheless interesting is that the GR law with the experimentally observed exponent is obtained for $\varepsilon \approx 0.5$ which is precisely the value for which the average threshold changes from a finite limit in the long term to a growing average. This could perhaps be connected to some kind of "self-organization" in the system, but this point is presently open.

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[1] B. Guttenberg and C. F. Richter, Ann. Geophys. **9**, 1 (1956); *Seismicity of the Earth and Associated Phenomena*, 2nd ed. (Princeton University Press, Princeton, NJ, 1954).
[2] J. B. Rundle *et al.*, Rev. Geophys. **41**, 1019 (2003).
[3] L. Gulia and S. Wiener, Geophys. Res. Lett. **37**, L10305 (2010).
[4] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).

[5] J. B. Rundle and D. D. Jackson, Bull. Seismol. Soc. Am. **67**, 1363 (1977).
[6] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. A **40**, 6470 (1989).
[7] Z. Olami, Hans Jacob S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
[8] K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).
[9] J. E. S. Socolar, G. Grinstein, and C. Jayaprakash, Phys. Rev. E **47**, 2366 (1993).
[10] P. Grassberger, Phys. Rev. E **49**, 2436 (1994).
[11] C. A. Serino, K. F. Tiampo, and W. Klein, Phys. Rev. Lett. **106**, 108501 (2011).
[12] S. Hainzl, G. Zöller, and J. Kurths, J. Geophys. Res. **104**, 7243 (1999); Nonlin. Proc. Geophys. **7**, 21 (2000).
[13] E. A. Jagla, Phys. Rev. E **81**, 046117 (2010).
[14] O. M. Braun and M. Peyrard, Phys. Rev. Lett. **100**, 125501 (2008).
[15] O. M. Braun and M. Peyrard, Phys. Rev. E **82**, 036117 (2010).
[16] Y. Ben-Zion and J. R. Rice, J. Geophys. Res. **100**, 12959 (1995).
[17] B. N. J. Persson, Phys. Rev. B **51**, 13568 (1995).
[18] A. E. Filippov, J. Klafter, and M. Urbakh, Phys. Rev. Lett. **92**, 135503 (2004).
[19] O. M. Braun and A. G. Naumovets, Surf. Sci. Rep. **60**, 79 (2006).
[20] O. M. Braun, Tribol. Lett. **39**, 283 (2010).
[21] O. M. Braun and E. Tosatti, Europhys. Lett. **88**, 48003 (2009).
[22] O. M. Braun and J. Röder, Phys. Rev. Lett. **88**, 096102 (2002).
[23] O. M. Braun, I. Barel, and M. Urbakh, Phys. Rev. Lett. **103**, 194301 (2009).
[24] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer, Berlin, 2004).
[25] O. M. Braun and M. Peyrard, Phys. Rev. E **83**, 046129 (2011).